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Painlevé Analysis and Some Solutions of (2 + 1)-Dimensional Generalized Burgers Equations

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Abstract *Burgers equation $u_t = 2uu_x + u_{xx}$ describes a lot of phenomena in physics fields, and it has attracted much attention. In this paper, the Burgers equation is generalized to $(2+1)$ dimensions. By means of the Painlevé analysis, the most generalized Painlevé integrable (2+1)-dimensional integrable Burgers systems are obtained. Some exact solutions of the generalized Burgers system are obtained via variable separation approach.*

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1 Introduction

The Burgers equation $u_t = 2uu_x + u_{xx}$ has attracted much attention since it was first proposed by Bateman.[1] Then Burgers[2] gave its some special solutions in 1940. Later on, Cole and Hopf[3,4] independently pointed out that any of the solutions of the heat equation $\xi_t = \xi_{xx}$ can be mapped to a solution of the Burgers equation. Recently, the Painleve property (PP) has shown its usefulness in studying of partial differential equation (PDE), especially in the sense of integrability. Some connections of PP with inverse scattering transform (IST) were found. Through Painleve analysis for an integrable model, we could probably find its various interesting properties like the Lax pairs.[5−7] In this paper, we would like to find $(2+1)$-dimensional Burgers equations which are Painleve integrable, and then use the variable separation approach to get some of their solutions.

2 Generalized Burgers Equations

The most generalized Burgers equation may have the form

$$\begin{aligned}\frac{\partial}{\partial t}u = {}& a_1 u\frac{\partial}{\partial x}u + a_2\frac{\partial^2}{\partial x^2}u + a_3 u\frac{\partial}{\partial y}u + a_4\frac{\partial^2}{\partial y^2}u \\ & + a_5\frac{\partial}{\partial x}u + a_6\frac{\partial}{\partial y}u + a_7 u + a_8\frac{\partial^2}{\partial x\partial y}u \\ & + b_1 v\frac{\partial}{\partial x}v + b_2\frac{\partial^2}{\partial x^2}v + b_3 v\frac{\partial}{\partial y}v(x,y,t) \\ & + b_4\frac{\partial^2}{\partial y^2}v + b_5\frac{\partial}{\partial x}v + b_6\frac{\partial}{\partial y}v + b_7 v \\ & + b_8\frac{\partial^2}{\partial x\partial y}v + c_1 u v_y + c_2 v u_x + c_3 uv\,, \end{aligned} \tag{1}$$

$$\frac{\partial}{\partial y}u = \frac{\partial}{\partial x}v\,, \tag{2}$$

where a_i, b_i, and c_i are all constants, which means that we are finding constant coefficient integrable models of Burger's type.

To check whether the given equations have Painleve property, we have several choices, such as ARS (Ablowitz–Ramani–Segur) algorithm,[8] the WTC (Weiss–Tabor–Carnevale) approach,[9] Kruskal's simplification,[10] the Conte's invariant method,[5] Pickering's approach,[7] and Lou's extended method.[11] Here we used WTC approach to analyse the equations.

As usual, we use a simple three-step version to find the possible models which have PP.

(i) Leading order analysis. Letting

$$u = \phi^{\alpha}\sum_{j=0}^{\infty} u_j\phi^j\,, \tag{3}$$

$$v = \phi^{\beta}\sum_{j=0}^{\infty} v_j\phi^j\,, \tag{4}$$

and substituting them into Eqs. (1) and (2) and balancing the leading nonlinear and leading dispersive terms, we could easily get $\alpha = \beta = -1$.

(ii) To find the resonance points. Because we are treating them as the Burger's type equation, we suppose that both a_1 and a_2 are not equal to zero. And the generalized (2+1)-dimensional Burgers equations have the resonance points which the original Burgers equation $u_t = 2uu_x + u_{xx}$ has. That means the generalized Burgers equations must have the resonance points at $j = 2$. Using the computer algebra such as Maple, we found that there are several cases in which the generalized (2+1)-dimensional Burgers equations are Painleve integrable. And the resonance points are $j = 2$, $j = 1$ and $j = -1$. Because of the complexity of the results, we do not write them down here.

(iii) To check if the compatibility at the resonance points holds, by which we mean that u_1, u_2 and ϕ must be arbitrary.

After finishing the above three steps and using rescaling procedure, we get the final nontrivial generalized (2+1)-dimensional Painleve integrable Burgers equations

$$u_t = a_1 uu_x + a_2 u_{xx} + b_1 vv_x + \frac{a_2 b_1}{a_1}v_{xy}\,, \quad u_y = v_x\,. \tag{5}$$

3 Some Solutions of Generalized Burgers Equations

Now we try to find some solutions of Eq. (5) through the variable separation approach.[11] We rewrite Eq. (5) as

$$u_t = A_1 uu_x + A_2 u_{xx} + B_1 vv_x + \frac{A_2 B_1}{A_1}v_{xy}\,,$$

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$$u_y = v_x \,. \tag{6}$$

The first step of the approach is to suppose that u and v take the following forms

$$u = 2(A_2/A_1)\ln(f)_x + u_0\,,\quad v = 2(A_2/A_1)\ln(f)_y + v_0\,, \tag{7}$$

which can be found from the truncated Painlevé expansion, where u_0, v_0 are arbitrary known solutions of Eqs. (6). Here, however, we take $u_0 = 0$ and $v_0 = v_0(y,t)$ for simplicity. The function v_0 is an arbitrary function of the indicated variables.

Then we suppose that f has the form

$$f = 1 + a_1 p(x,t) + a_2 q(y,t) + Ap(x,t)q(y,t)\,, \tag{8}$$

where a_1, a_2 and A are constants to be determined; $p(x,t)$ and $q(y,t)$ are at present supposed to be arbitrary. Substituting Eqs. (7) with Eq. (8) into Eqs. (6) yields

$$\begin{aligned} 0 &= 2A_2(-A_1A^2p_xq^2 - A_1a_1^2p_x - 2A_1a_1p_xAq)p_t + 2A_2(A_1Ap_x - a_1A_1p_xa_2)q_t \\ &\quad + 2A_2(2a_1A_1Aqp + A_1Aq^2a_2 + A_1a_1^2p + a_1A_1 + a_1A_1a_2q + A_1Aq + A_1A^2q^2p)p_{xt} \\ &\quad + 2A_2(a_2A_2A_1Aq^2 + A_2A_1A^2q^2p + 2A_2a_1A_1Apq + a_2a_1A_1 + a_2A_2a_1A_1q + A_2A_1Aq + A_2A_1a_1^2p)p_{xxx} \\ &\quad + 2A_2(-A_2A_1A^2p_xq^2 - A_2A_1a_1^2p_x - 2A_2a_1A_1p_xAq)p_{xx} \\ &\quad + 2A_2(-A_2a_2B_1a_1p_x + A_2B_1Ap_x)q_{yy} + 2A_2(-B_1A_1a_1a_2p_xq_y + B_1A_1Ap_xq_y)v_0\,. \end{aligned} \tag{9}$$

Using the computer algebras such as Maple or Mathematica, we get

$$\begin{aligned} 0 &= 2JA_1A_2(-Jp_x + f\partial_x)(p_t + A_2p_{xx}) \\ &\quad - 2A_2Kp_x(A_2B_1q_{yy} + A_1q_t + A_1B_1q_yv_0)\,, \end{aligned} \tag{10}$$

where $J = Aq + a_1$ and $K = a_1a_2 - A$. Because p is y-independent and q is x-independent, equation (10) can be separated into two parts,

$$p_t + A_2p_{xx} = d_2p^2 + d_1p + d_0\,, \tag{11}$$

$$A_2B_1q_{yy} + A_1q_t + A_1B_1q_yv_0 = c_2q^2 + c_1q + c_0\,. \tag{12}$$

Substituting Eqs. (11) and (12) into Eq. (10), we can determine the values of c_i and d_i. There are two cases. The first one is

$$d_2 = 0\,,\quad d_0 = d_1/a_1\,,\quad A = a_1a_2 \tag{13}$$

with a_1, a_2, $c_0(y,t)$, $c_1(y,t)$, $c_2(y,t)$, and $d_1(t)$ being arbitrary constants/functions with indicated variables. The second one is

$$\begin{aligned} &d_2 = 0\,,\qquad c_1 = (2a_1c_2 - A_1Ad_1)/A\,, \\ &d_0 = (A_1Aa_2d_1 - a_1a_2c_2 + Ac_2)/A_1A^2\,, \\ &c_0 = a_1(a_1c_2 - A_1Ad_1)/A^2 \end{aligned} \tag{14}$$

with A, a_1, a_2, and $c_2(t)$ and $d_1(t)$ being arbitrary.

Both $v_0(y,t)$ and $q(y,t)$ are previously supposed to be arbitrary of the indicated variables. But from Eq. (12) we know that we must fix one of them, say v_0, and let the other be free. Hence, we obtain a solution of the generalized Burgers equation (6) with $q(y,t)$ being arbitrary

$$u = \frac{2A_2}{A_1}\frac{p_x(a_1 + Aq)}{(1 + a_1p + a_2q + Apq)}\,,$$

$$v = \frac{2A_2}{A_1}\frac{q_y(a_2 + Ap)}{(1 + a_1p + a_2q + Apq)} + v_0\,, \tag{15}$$

where p must satisfy Eq. (11) while v_0 is determined by $q(y,t)$ and c_i $(i = 0,1,2)$ according to Eq. (12).

4 Summary and Discussion

In summary, using the WTC method, we obtained the most generalized $(2+1)$-dimensional constant coefficients Burgers equations which have Painleve̒ property (Painleve̒ integrable). Then with the help of Lou's separation method,[12] we get solutions with some arbitrary functions. As in the articles,[12] by fixing these arbitrary functions, we could get some interesting special solutions. We leave it undone because of two points. First, there are complete details in how to select functions to get interesting different solutions, which show the probable connections between non-integrable systems and integrable systems. Second, those who are at liberation know better how to fix the arbitrary functions to explain phenomena in experiments.

The next question is what type of partial differential equations can be separated in Lou's method to get solutions with arbitrary functions. This is investigated now by a group under professor S.Y. Lou. Another question is why those equations can be separated in this way. What is the mathematical or physical reasons? It deserves consideration.

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